

Latin trades and simplicial complexes

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Abstract

In this note we introduce the concept of the *trade space* of a latin square. Computations using Sage and the GAP package Simplicial Homology are presented.

1 Introduction

We first introduce the concept of a latin square and a latin bitrade. For a more detailed exposition and literature survey, see [8].

Definition 1.1. Let N be a fixed set of size $n > 0$. A *latin square* L of order n is an $n \times n$ array with rows and columns indexed by N , and entries from the set N . Further, each $e \in N$ appears exactly once in each row and exactly once in each column. A *partial latin square* of order n is an $n \times n$ array where each $e \in N$ occurs at most once in each row and at most once in each column.

Usually for index set N we will use $[n] = \{1, 2, \dots, n\}$ or sometimes $[n]_0 = \{0, 1, \dots, n-1\}$ when working with modulo arithmetic. Note that a latin square is a partial latin square with no empty cells. A latin square L may also be represented as a set of ordered triples, where $(r, c, e) \in L$ denotes the fact that symbol e appears in the cell at row r , column c , of L . Alternatively we write L^\diamond with binary operation \diamond such that $r \diamond c = e$ if and only if $(r, c, e) \in L$. Similarly, a partial latin square P may be written as P^\diamond . We use setwise and binary operator notation interchangeably.

The *size* of a partial latin square P is the number of filled cells, denoted by $|P| = |\{(r, c, e) \mid (r, c, e) \in P\}|$. A partial latin square P' is *contained in*, or is a (*partial*) *subsquare* of P if and only if $P' \subseteq P$.

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1.1 Latin bitrades and trades

Definition 1.2. A *latin bitrade* (T^\diamond, T^\otimes) is a pair of partial latin squares such that:

1. $\{(i, j) \mid (i, j, k) \in T^\diamond \text{ for some symbol } k\}$
 $= \{(i, j) \mid (i, j, k') \in T^\otimes \text{ for some symbol } k'\};$
2. for each $(i, j, k) \in T^\diamond$ and $(i, j, k') \in T^\otimes$, $k \neq k'$;
3. the symbols appearing in row i of T^\diamond are the same as those of row i of T^\otimes ;
the symbols appearing in column j of T^\diamond are the same as those of column j of T^\otimes .

The following definition is equivalent to Definition 1.2:

Definition 1.3. A *latin bitrade* (T^\diamond, T^\otimes) is a pair of partial latin squares $T^\diamond, T^\otimes \subseteq A_1 \times A_2 \times A_3$ such that:

- (R1) $T^\diamond \cap T^\otimes = \emptyset$;
- (R2) for all $(a_1, a_2, a_3) \in T^\diamond$ and all $r, s \in \{1, 2, 3\}$, $r \neq s$, there exists a unique $(b_1, b_2, b_3) \in T^\otimes$ such that $a_r = b_r$ and $a_s = b_s$;
- (R3) for all $(a_1, a_2, a_3) \in T^\otimes$ and all $r, s \in \{1, 2, 3\}$, $r \neq s$, there exists a unique $(b_1, b_2, b_3) \in T^\diamond$ such that $a_r = b_r$ and $a_s = b_s$.

Note that (R2) and (R3) imply that each row (column) of T^\diamond contains the same subset of A_3 as the corresponding row (column) of T^\otimes . Since all of the bitrades in this dissertation are latin bitrades, we usually shorten ‘latin bitrade’ to just ‘bitrade.’

For a bitrade (T^\diamond, T^\otimes) we refer to T^\diamond as the *trade*, and T^\otimes as the *disjoint mate*. A particular trade may have more than one disjoint mate.

Example 1.4. Consider the following latin squares:

$$\begin{array}{c}
\begin{array}{c|cccccc}
\diamond & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 1 & 3 & 2 & 4 & 5 \\
1 & 5 & 2 & 4 & 3 & 1 & 0 \\
2 & 2 & 3 & 5 & 4 & 0 & 1 \\
3 & 3 & 4 & 1 & 0 & 5 & 2 \\
4 & 4 & 5 & 0 & 1 & 2 & 3 \\
5 & 1 & 0 & 2 & 5 & 3 & 4
\end{array}
\quad
\begin{array}{c|cccccc}
\otimes & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 3 & 1 & 2 & 4 & 5 \\
1 & 5 & 2 & 3 & 4 & 1 & 0 \\
2 & 2 & 4 & 5 & 3 & 0 & 1 \\
3 & 3 & 1 & 4 & 0 & 5 & 2 \\
4 & 4 & 5 & 0 & 1 & 2 & 3 \\
5 & 1 & 0 & 2 & 5 & 3 & 4
\end{array}
\end{array}
\quad (1)$$

One possible bitrade (T^\diamond, T^\otimes) where $T^\diamond \subseteq L^\diamond$ and $T^\otimes \subseteq L^\otimes$ is shown below:

$$\begin{array}{c}
\begin{array}{c|cccccc}
\diamond & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & & 1 & 3 & & & \\
1 & & & 4 & 3 & & \\
2 & & 3 & & 4 & & \\
3 & & 4 & 1 & & & \\
4 & & & & & & \\
5 & & & & & &
\end{array}
\quad
\begin{array}{c|cccccc}
\otimes & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & & 3 & 1 & & & \\
1 & & & 3 & 4 & & \\
2 & & 4 & & 3 & & \\
3 & & 1 & 4 & & & \\
4 & & & & & & \\
5 & & & & & &
\end{array}
\end{array}
\quad (2)$$

So $L^\otimes = (L^\diamond \setminus T^\diamond) \cup T^\otimes$ and $L^\diamond = (L^\otimes \setminus T^\otimes) \cup T^\diamond$.

Example 1.5. The following bitrade is simply a cyclic row-shift of $T^\diamond = \mathbb{Z}_3$, the integers under addition modulo 3:

$$T^\diamond = \begin{array}{c|ccc} \diamond & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 1 \end{array} \quad T^\otimes = \begin{array}{c|ccc} \otimes & 0 & 1 & 2 \\ \hline 0 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 \end{array}$$

Example 1.6. Here is a larger bitrade:

$$\begin{array}{c|cccc} \diamond & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 4 & 3 & 2 \\ 2 & 4 & 3 & & \\ 3 & & & 2 & 1 \\ 4 & 3 & & 1 & \end{array} \quad \begin{array}{c|cccc} \otimes & 1 & 2 & 3 & 4 \\ \hline 1 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & & \\ 3 & & & 1 & 2 \\ 4 & 1 & & 3 & \end{array} \quad (3)$$

1.2 Latin critical sets

Definition 1.7. A partial latin square $C \subseteq L$ is a *critical set* if

1. C has unique completion to L ; and
2. no proper subset of C satisfies 1.

Example 1.8. Latin square \mathbb{Z}_2 and a critical set:

0	1
1	0

	0

Example 1.9. Latin square L_3 and critical set $P_3 \subset L_3$:

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

0	1	2	3	4	5	6	
1	0	3	2	5	4		
2	3	0	1	6		4	
3	2	1	0				
4	5	6		0	1	2	
5	4			1	0		
6		4		2		0	

Fix a latin square L and define the *trade space*

$$\mathcal{T}_L = \{T^\diamond \mid T^\diamond \subset L \text{ and } (T^\diamond, T^\otimes) \text{ is a bitrade}\}.$$

Lemma 1.10. Let C be a critical set of the latin square L . Then for any $T \in \mathcal{T}_L$, there exists $c \in C$ such that $c \in T$.

An long-standing open conjecture is that the size of the smallest critical set in any latin square of order n is $\lfloor n^2/4 \rfloor$. See [2], [4] and [7] for recent work on this problem.

1.3 Simplicial complexes

We follow the presentation of Faridi [6]. A *facet* of a simplicial complex is a maximal face under inclusion. A *vertex cover* A of a simplicial complex Δ is a subset of vertices of Δ such that any facet is adjacent to some $a \in A$. A *minimal vertex cover* A is a vertex cover of Δ such that no proper subset of A is a vertex cover.

Let L be a latin square of order n . Form the simplicial complex $\Delta(L)$ with vertices corresponding to trades $T \subset L$, and edges (T, T') for trades $T, T' \subset L$ such that $T \cap T' \neq \emptyset$, and so on for faces of higher dimensions.

Lemma 1.11. *For each critical set C of L there exists a minimal vertex cover A_C of $\Delta(L)$ and $|C| = |A_C|$.*

Proof. Let $C = \{c_1, \dots, c_m\}$ be a critical set in L . Create a family of m sets

$$S_i = \{T \subseteq L \mid c_i \in T\}$$

where T is a trade in L . We will show that the S_i have a system of distinct representatives. Choose any ℓ of the sets, $S_{i_1}, \dots, S_{i_\ell}$ and suppose that their union \mathcal{S} contains $l < \ell$ trades. By the pigeonhole principle there is an entry c_{i_u} such that each trade that intersects c_{i_u} also intersects some c_{i_v} , $v \neq u$. So the set $C \setminus \{c_{i_u}\}$ is a critical set, contradicting the minimality of C . Call the system of distinct representatives A_C .

To see that A_C is a vertex cover of $\Delta(L)$, let F be some facet of $\Delta(L)$. Pick any $T \in F$. Then there exists some $c_i \in C$ such that $c_i \in T$. We chose some \bar{T} as the representative of S_i . There is an edge (T, \bar{T}) due to the common element c_i , so F is adjacent to $\bar{T} \in A_C$.

For minimality, suppose that $A_C \setminus \{T\}$ is a vertex cover for some trade T in L . Suppose that T was chosen as the representative of S_i . Let U be any trade in L and consider $C' = C \setminus \{c_i\}$. If $c_i \notin U$ then some other entry of C covers U since C is a critical set. Otherwise, suppose that $c_i \in U$ then consider the facet F containing T and U . By assumption there is some $T' \in A_C \setminus \{T\}$ such that $T' \in F$. Then T' must be the representative for S_j (where $j \neq i$ since A_C is a system of distinct representatives) so $c_j \in U$. In this way any trade U is covered by some element of $C' = C \setminus \{c_i\}$, contradicting the minimality of C . \square

Lemma 1.12. *For each minimal vertex cover A of $\Delta(L)$ there exists a critical set C_A of L such that $|C_A| = |A|$.*

Proof. Let A be a minimal vertex cover of $\Delta(L)$. For each $T \in A$, let x_T be an entry of T contained in the intersection of the trades that make up the facet containing T . We show that X is a critical set. Suppose that U is some trade in L and no x_T is in U . The vertex U is adjacent to some facet F and by definition some $T' \in A$ is adjacent to this F . Then by definition $x_{T'} \in U$, a contradiction. Thus X is a uniquely completable partial latin square.

For minimality, suppose that the set $X' = X \setminus \{x_T\}$ is uniquely completable for some $x_T \in X$. By the previous lemma, X' is equivalent to a vertex cover

A' of the same size, contradicting the minimality of A . So X is minimal and $|X| = |A|$. \square

Remark 1.13. A critical set of L is equivalent to a vertex cover of $\Delta(L)$. The simplicial complex $\Delta(L)$ captures information about the intersections of all trades. The critical sets correspond to minimal vertex covers.

Faridi's paper ([6], Proposition 1) shows that a minimum vertex cover is the same as a minimal prime ideal in the *facet ideal* $\mathcal{F}(\Delta)$ in the polynomial ring $R = k[x_1, \dots, x_n]$. In future research we would like to calculate more detailed information about the facet ideal and its minimal prime ideals.

2 Computations

In this section we present computations of the *reduced homology groups* H_k of the facet ideal $\Delta(L)$ for various latin squares L of small order. Informally, the group H_k indicates the number of k -dimension 'holes' in the simplicial complex at hand. See [9] for more information about homology theory. We use a Sage script [1] and the GAP package Simplicial Homology to calculate the reduced homology groups. For more information about Sage see [10]. The Simplicial Homology package is available at <http://www.cis.udel.edu/~dumas/Homology/>.

In this section, B_n denotes the addition table for integers modulo n (also known as the back-circulant square). Also, let L_s denote the latin square corresponding to the group table of the elementary abelian 2-group of order 2^s . Formally, these latin squares are defined by

$$L_1 = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}$$

and for $s \geq 2$,

$$L_s = L_1 \times L_{s-1} = \{(x, y; z), (x, y + n/2; z + n/2), (x + n/2, y; z + n/2), \\ (x + n/2, y + n/2; z) \mid (x, y; z) \in L_{s-1}\}$$

For example,

$$L_3 = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ \hline 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ \hline 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ \hline 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ \hline 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ \hline 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array}$$

We now present computations for small orders:

Latin square	Homology groups
B_3	$[0, 0, 0, 10, 0, 0]$
B_4	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1]$
L_2	$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 8, 9, 0]$

So for B_3 we have $\tilde{H}_0 = 0$, $\tilde{H}_1 = 0$, $\tilde{H}_2 = 0$, $\tilde{H}_3 = 10$, $\tilde{H}_4 = 0$, $\tilde{H}_5 = 0$.

2.1 Intercalate homology

An *intercalate* is a latin trade of size four. Intercalates are interesting because they are the simplest (smallest) type of latin trade and in fact any trade can be written as a sum of intercalates [5]. Another interesting question is whether there are latin squares with as many intercalates as possible, or as few as is possible (see for example [3]).

For a latin square L we define the *intercalate trade space* as

$$\mathcal{I}_L = \{T^\diamond \mid T^\diamond \subset L, |T^\diamond| = 4 \text{ and } (T^\diamond, T^\otimes) \text{ is a bitrade}\}.$$

We can then create a simplicial complex where vertices are intercalates, and higher dimension simplices are collections intercalates that intersect in some common point. Here is the homology group information for the elementary abelian 2-group of various orders:

Latin square	Intercalate homology groups
L_1	$[0, 0, 0, 0]$
L_2	$[0, 21, 0, 0]$
L_3	$[0, 273, 0, 0]$
L_4	$[0, 2625, 0, 0]$
L_5	$[0, 22785, 0, 0]$
L_6	$[0, 189441, 0, 0]$

For the back-circulant B_n with $n \geq 4$ and n even we have $\tilde{H}_1 = 0$, $\tilde{H}_2 = 0$, $\tilde{H}_3 = 0$, and only $\tilde{H}_0 \neq 0$. The values for \tilde{H}_0 are given below:

3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143, 168, 195, 224, 255, 288, 323, 360, 399, 440, 483, 528, 575, 624, 675, 728, 783, 840, 899, 960, 1023, 1088, 1155, 1224, 1295, 1368, 1443, 1520, 1599, 1680, 1763, 1848, 1935, 2024, 2115, 2208, 2303, 2400, 2499, 2600, 2703, 2808, 2915, 3024, 3135, 3248, 3363, 3480, 3599, 3720, 3843, 3968, 4095, 4224, 4355, 4488, 4623, 4760, 4899, 5040, 5183, 5328, 5475, 5624, 5775, 5928, 6083, 6240, 6399, 6560, 6723, 6888, 7055, 7224, 7395, 7568, 7743, 7920, 8099, 8280, 8463, 8648, 8835, 9024, 9215, 9408, 9603, 9800.

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